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Resource Dynamics

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1. Introduction

Our starting point in this paper is a previously developed model for a dynamic optimal budgeting problem that was stated, along with several related models, by Clark (1983) and analyzed by Falk and McCormick (1983). It is model 2.0 in the nomenclature of Clark (1983), and it seeks to maximize a measure of the effective asset value of owned resources. These resources change over time from period to period according to specified relations and the amounts of procurement and of maintenance and manpower supplied.

Upon observing that the relations connecting variables in adjacent time periods are nearly independent of the specific time periods, we thought it of interest to examine a simpler model, in which they are *completely* independent of the specific time periods, and to analyze this model under the assumption that steady-state values are obtained as the number of time periods increases without limit. Such analysis constitutes the main part of this paper.

An outline of the paper is as follows. In Section 2 we present the dynamic model and briefly explain its variables and the relations between them; this is done in order to make the present work self-contained. In Section 3 we derive and justify the steady-state model, henceforth called model 2.1, and in Section 4 we simplify and condense it in order to essentially solve it in closed form. Sections 5 and 6 analyze model 2.1 with two different objective functions, leading to models 2.1(A) and 2.1(B), respectively, and show how to numerically obtain optimal solutions. Optimal solutions are provided for both the input data of Clark (1983) and the (slightly different) data of Falk and McCormick (1983). The final section offers some tentative conclusions and suggests several directions for further study.

2. The Dynamic Model

The dynamic model is composed of state variables, control variables, and the relations between them. There are also initial values of the state variables and numerical parameters to be treated as data. The state and control variables are as follows.

State variables:

B_t	budget in period t
P_t	procurement in period t
MX_t	maintenance demanded in period t
MXS_t	maintenance supplied in period t
$MBLOG_t$	maintenance backlog in period t
M_t	manpower demanded in period t

MS_t manpower supplied in period t

AV_t asset value in period t

R_t asset value retired in period t

EAV_t effective asset value in period t

Control variables:

F_{1t} fraction of maintenance supplied to total maintenance demanded, in period t

F_{2t} fraction of manpower supplied to manpower demanded, in period t.

F_{1t} and F_{2t} are restricted to the interval $[0,1]$, while the state variables are only required to be nonnegative. The index t takes values $1,2,\ldots$.

We now present the relations that link the variables in various time periods. To the right of each relation is an appropriate explanatory phrase, but some additional explanation will also be given below. In examining these relations, one should note that they are all definitional in nature, as opposed to expressing constraints on the variables. Indeed, the only real constraints on the variables are those simple ones stated above.

Relations of model 2.0:

$B_t = B_{02}(1 + a_1)^t$	Budget in period t	(1)
$M_t = ca_6(1 + a_5)^t AV_{t-1}$	Manpower demanded	(2)
$MS_t = (F_{2t})\, M_t$	Manpower supplied	(3)

$$MX_t = a_4(AV_{t-1} - a_7 MBLOG_{t-1}) \qquad \text{Maintenance demanded} \qquad (4)$$

$$MXS_t = (F_{1t})(MBLOG_{t-1} + MX_t) \qquad \text{Maintenance supplied} \qquad (5)$$

$$MBLOG_t = (a_3)^{a_2} MBLOG_{t-1} + MX_t - MXS_t \qquad \text{Maintenance backlog} \qquad (6)$$

$$P_t = B_t - MXS_t - MS_t \qquad \text{Procurement} \qquad (7)$$

$$R_t = (1 - a_3)AV_{t-1} \qquad \text{Retiring asset value} \qquad (8)$$

$$AV_t = AV_{t-1} + P_t - R_t \qquad \text{Asset value.} \qquad (9)$$

Relation (1) shows the budget to be growing at a rate of a_1; B_{02} is the initial budget. Relation (2) shows manpower demanded to be directly proportional to asset value in the previous period, while (3) indicates that manpower supplied is simply some fraction of that demanded.

Relations (4) and (5) are similar to (2) and (3), respectively, but for maintenance demanded and supplied; the relationships are a bit more complex in that the maintenance backlog also enters into them. Relation (6) updates the maintenance backlog.

Relation (7) defines procurement as that amount left after manpower and maintenance have been supplied, and (8) specifies the asset value retired as a given fraction of asset value in the previous period. Relation (9) updates the asset value.

Before further discussion of this model, it is appropriate here to list the currently estimated values of the parameters and initial values that appear in (1) - (9), as supplied by Clark (1983).

<u>Parameter values</u>:

$a_1 = 0.03$

$a_2 = 2.0$

$a_3 = 0.95$

$a_4 = 0.04$

$a_5 = -0.01$

$a_6 = 25 \times 10^{-6}$

$a_7 = 0.5$

$c = 1200$

<u>Initial values</u>:

$B_{02} = 10$

$AV_0 = 100$

$MBLOG_0 = 1$

Falk and McCormick (1983) used an earlier estimated set of value which, however, differs only in using $a_3 = 29/30$ and $c = 1000$.

The relations (1) - (9) can be greatly condensed through simple substitution and elimination; e.g., B_t in (7) can be replaced by the right-hand side of (1), M_t in (3) can be replaced by the right-hand side of (2), etc. Equally important, the control variables F_{1t} and F_{2t} can be eliminated by using MS_t and MXS_t as control variables and replacing relations (3) and (5) by the following two:

$$MS_t \leqslant M_t \tag{3'}$$

$$MXS_t \leqslant MBLOG_t + MX_t \quad . \tag{5'}$$

This is justified by the requirement that all variables be nonnegative.

Falk and McCormick (1983) have used this approach to reduce (1) - (9) to a set of four linear constraints involving nonnegative variables. They then considered a particular objective function, one we treat in Section 6, and completely solved an approximate version of model 2.0 covering five time periods. We will also use this approach, but only after passing to a steady-state version of model 2.0; this is the subject of the next section.

3. The Steady-state Model

Examination of relations (1) - (9), with either (3) and (5) or (3') and (5'), shows that there is very little dependence on the actual value of t (as opposed to the change from $t-1$ to t); it exists only in relations (1) and (2), by virtue of the terms $(1 + a_1)^t$ and $(1 + a_5)^t$, respectively. Since the values of a_1 and a_5 are rather small in magnitude (the current estimates are $a_1 = 0.03$ and $a_5 = -0.01$), over a fairly short time horizon relations (1) and (2) are nearly independent of the actual value of t. It is therefore of interest to examine a version of model 2.0 in which $a_1 = a_5 = 0$ so that the recursion relations are independent of the actual time periods.

Under the assumption $a_1 = a_5 = 0$, (1) reduces to $B_t = B_{02}$ for all t and (2) becomes $M_t = c_6 AV_{t-1}$, where $c_6 \equiv ca_6$. Relation (1) is now redundant, and we are left with the following:

$$M_t \quad = c_6 AV_{t-1} \tag{10}$$

$$MS_t \leqslant M_t \tag{11}$$

$$MX_t = a_4 AV_{t-1} - a_4 a_7 MBLOG_{t-1} \tag{12}$$

$$MXS_t \leqslant MBLOG_t + MX_t \tag{13}$$

$$MBLOG_t = (a_3)^{a_2} MBLOG_{t-1} + MX_t - MXS_t \tag{14}$$

$$P_t = B_{02} - MXS_t - MS_t \tag{15}$$

$$R_t = (1 - a_3) AV_{t-1} \tag{16}$$

$$AV_t = AV_{t-1} + P_t - R_t \tag{17}$$

Now suppose we append to model 2.0 an objective function of the form

$$\sum_{t=1}^{T} v_t ,$$

where v_t is a measure of effectiveness in period t . It is certainly equivalent to consider the average value objective function

$$V(T) \equiv \frac{1}{T} \sum_{t=1}^{T} v_t . \tag{18}$$

We now make the additional assumption that v_t has the same algebraic form for all t and depends on t through its dependence on the values of the state and control variables in period t . It then seems likely that the optimal value of $V(T)$ approaches a limit as T approaches infinity and, more importantly, that the optimal values of the control and state variables also approach limits as T approaches infinity. Assuming the existence of such limiting optimal values, we denote them by simply omitting the subscript t (we also set $B \equiv B_{02}$). These limiting values then satisfy relations obtained from (10) - (17)

by omitting the subscripts t and t-1 . We thus obtain the following relations:

$$M = c_6 AV \tag{19}$$

$$MS \leqslant M \tag{20}$$

$$MX = a_4 AV - a_4 a_7 MBLOG \tag{21}$$

$$MXS \leqslant MBLOG + MX \tag{22}$$

$$MBLOG = (a_3)^{a_2} MBLOG + MX - MXS \tag{23}$$

$$P = B - MXS - MS \tag{24}$$

$$R = (1 - a_3)AV \tag{25}$$

$$AV = AV + P - R \tag{26}$$

$$AV, M, MS, MX, MXS, MBLOG, P, R \geqslant 0 , \tag{27}$$

where $B > 0$ is given.

We refer to the model specified by (19) - (27) as (steady-state) model 2.1; we proceed to analyze it below. Note first, however, that our rationale for examination of model 2.1, especially with an appropriate objective function, is to gain insight into the nature of optimal solutions when relations (10) - (17) are used, which in turn approximate (dynamic) model 2.0.

4. Closed-form Solution of the Steady-state Model

In this section we greatly simplify model 2.1 through substitution and condensation, much in the way that Falk and McCormick (1983) simplified model 2.0. We are able to go significantly further, however, because of the fact that model 2.1 is time-independent.

We first replace M in (20) by c_6AV and eliminate (19). We also replace R in (26) by $\bar{a}_3AV$, where $\bar{a}_3 \equiv 1 - a_3$, and eliminate (25). Then (26) yields $P = R = \bar{a}_3AV$, so we replace P in (24) by $\bar{a}_3AV$ and eliminate (26). Solving (23) for MX, we obtain

$$MX = MXS + a_{23}MBLOG , \tag{28}$$

where $a_{23} \equiv 1 - (a_3)^{a_2} > 0$. Thus we eliminate (23) and substitute the right-hand side of (28) for MX into (21) and (22), yielding, respectively,

$$MXS = a_4AV - a_{27}MBLOG , \tag{29}$$

$$MXS \leqslant (1 + a_{23})MBLOG + MXS , \tag{30}$$

where $a_{27} \equiv a_4a_7 + a_{23}$. Relation (30) is clearly redundant, and so is eliminated. We now replace MXS in (24) by the right-hand side of (29) and add the constraint that the right-hand side of (29) must be nonnegative. We are then left with the following relations:

$$MS \leqslant M \quad \text{[from (20)]} \tag{31}$$

$$a_4AV - a_{27}MBLOG \geqslant 0 \quad \text{[from (29)]} \tag{32}$$

$$a_{34}AV + MS - a_{27}MBLOG = B \quad \text{[from (24)]} , \tag{33}$$

$$AV,\ M,\ MS,\ MBLOG \geqslant 0 , \tag{34}$$

where $a_{34} \equiv \bar{a}_3 + a_4$.

Now define the ratios

$$x_2 \equiv MBLOG/AV \tag{35}$$

$$x_3 \equiv MS/M . \tag{36}$$

Then (31) and (34) imply $0 \leqslant x_3 \leqslant 1$, and (32) and (34) imply

$0 \leqslant x_2 \leqslant a_4/a_{27}$. In (33) we now substitute $MS = Mx_3 = c_6AVx_3$ [by (19)] and $MBLOG = AVx_2$ to obtain

$$AV(a_{34} + c_6x_3 - a_{27}x_2) = B \ . \tag{37}$$

Solving (37) for AV , and appending the constraints on x_2 and x_3 , model 2.1 is now reduced to the following:

$$AV = B/(a_{34} - a_{27}x_2 + c_6x_3) \tag{38}$$

$$0 \leqslant x_2 \leqslant a_4/a_{27} \tag{39}$$

$$0 \leqslant x_3 \leqslant 1 \ . \tag{40}$$

Expression (38) gives AV in closed form as a function of x_2 and x_3 , our new control variables. The choice of x_2 and x_3 as control variables is motivated by the choices of effectiveness functions that we shall employ in the following two sections. In each case the task is to find the optimal values of x_2 and x_3 ; once these are determined the corresponding values of all state variables can be found from (38) and the relations given earlier. Thus relations (38) - (40) provide the starting point for the optimality analysis of steady-state model 2.1.

5. Optimality Analysis of Model 2.1(A)

In model 2.0(A) (the "A" denotes the first type of objective function) each function v_t is of the form

$$\begin{aligned} v_t &= w_t f_t(AV_t, MBLOG_t, M_t, MS_t) \\ &= w_t AV_t h_2(x_{2t}) h_3(x_{3t}) \ , \end{aligned} \tag{41}$$

where w_t is a positive weight,

$$x_{2t} \equiv MBLOG_t/AV_t ,$$

$$x_{3t} \equiv MS_t/M_t ,$$

h_2 is a strictly decreasing and continuous function on $[0,1]$ with $h_2(0) = 1$ and $h_2(1) = 0$, and h_3 is a strictly increasing and continuous function on $[0,1]$ with $h_3(0) = 0$ and $h_3(1) = 1$.

The rationale for the above choice is as follows. In each period t it is desired to have AV_t as large as possible, but if this is achieved by diverting funds away from needed maintenance and manpower demands, then the *effective* asset value, EAV_t, is less than AV_t. Expression (41) gives EAV_t as AV_t diminished through multiplication by two quantities between 0 and 1, h_2 and h_3, that account for degradation of useful asset value due to (1) a maintenance backlog, relative to asset value $[h_2(x_{2t})]$; and (2) a manpower shortage relative to manpower demanded $[h_3(x_{3t})]$.

The current choices for the degradation functions h_2 and h_3 are cumulative beta functions of the following form:

$$h_2(x_2) = (1 + p_2x_2)(1 - x_2)^{p_2} , \tag{42}$$

$$h_3(x_3) = x_3^{p_3}(1 + p_3 - p_3x_3) , \tag{43}$$

where p_2 and p_3 are parameters, currently estimated at $p_2 = 58$ and $p_3 = 5$.

Let us now insert v_t from (41) into the average value objective function (18); we obtain

$$V(T) = \frac{1}{T}\sum_{t=1}^{T} w_tAV_th_2(x_{2t})h_3(x_{3t}) . \tag{44}$$

Suppose that the positive weights w_t satisfy $\sum_{t=1}^{T} w_t = T$ and $\lim_{t\to\infty} w_t = 1$. Assuming that $AV_t \to AV$, $x_{2t} \to x_2$, and $x_{3t} \to x_3$ as $t \to \infty$, we see that as $T \to \infty$, $V(T)$ approaches the limit

$$V_A = V_A(x_2,x_3) \equiv AVh_h(x_2)h_3(x_3) \ . \tag{45}$$

In (45) it is appropriate to write $V_A(x_2,x_3)$, since (38) shows AV to depend solely on the control variables x_2 and x_3 .

Thus an appropriate objective function for steady state model 2.1 is $V_A(x_2,x_3)$, as given by (45), and so we henceforth refer to the following optimization problem as model 2.1(A):

$$\begin{aligned} &\text{Maximize} \quad V_A(x_2,x_3) = AV(x_2,x_3)h_2(x_2)h_3(x_3) \\ &\text{subject to} \quad 0 \leqslant x_2 \leqslant a_4/a_{27} \ , \\ &\qquad\qquad\qquad 0 \leqslant x_3 \leqslant 1 \ . \end{aligned}$$

Model 2.1(A) is a very small and relatively simple nonlinear programming (NLP) problem and might perhaps be best solved by traditional methods of NLP [see, e.g., McCormick (1983)]. Because of the simple structure of model 2.1(A), however, we have attempted to find a solution more directly. We have done this by maximizing instead the logarithm of $V_A(x_2,x_3)$, and to do this latter we have (partially) ignored the constraints (39) and (40) and set the two first partial derivatives of $\log V_A(x_2,x_3)$ to zero. We have used the specific forms for h_2 and h_3 given by (42) and (43).

Setting the two first partial derivatives of $\log V_A(x_2,x_3)$ to zero yields two quadratic equations involving x_2 and x_3 , so no closed-form solution is possible. We therefore solved the respective quadratic equations for each variable in terms of the other one, using

the interval constraints $0 \leqslant x_2 \leqslant a_4/a_{27}$ and $0 \leqslant x_3 \leqslant 1$ to select the appropriate root, and thereby obtained a rapidly convergent iterative algorithm. Regardless of starting point, it yields the optimal values of x_2 and x_3 to seven significant figures after only three iterations. It remains, actually, to check the boundary conditions of the interval constraints on x_2 and x_3, but this is easy to do. We have $h_3(0) = 0$ and $h_2(a_4/a_{27})$ is extremely small, so the only cases worthy of examination are those where $x_2 = 0$ and/or $x_3 = 1$. But But $\partial(\log V_A)/\partial x_2 > 0$ at $x_2 = 0$ and $\partial(\log V_A)/\partial x_3 < 0$ at $x_3 = 1$, so neither $x_2 = 0$ or $x_3 = 1$ can yield an optimal solution.

To verify that the iterative algorithm sketched above does indeed yield a local optimum, and hence a global optimum, we have verified numerically that the Hessian matrix of $\log V_A(x_2,x_3)$ is negative definite at the solution point found.

Table 1 presents the optimal solutions to model 2.1(A) for both the data sets of Clark (1983) and Falk and McCormick (1983). Note that in both cases the optimal values of x_2 and x_3 are extremely close to 0 and 1, respectively. Also shown in Table 1 are the values of $V_A(x_2 = 0,\ x_3 = 1)$; these are only about 0.1% less than the optimal values. For practical purposes, therefore, the optimal solution to model 2.1(A), with presently available data, occurs at $x_2 = 0$, $x_3 = 1$.

TABLE 1

SOLUTIONS OF MODEL 2.1(A)

Variable	Optimal Value	Near-optimal Value
(A) Data from Falk and McCormick (1983)		
x_2	0.000259	0.0
x_3	0.9912	1.0
AV	101.95	101.69
V_A	101.82	101.69
MBLOG	0.0264	0.0
MX	4.077	4.068
MXS	4.076	4.068
M	2.549	2.542
MS	2.526	2.542
P,R	3.398	3.390
B	10.0	10.0
(B) Data from Clark (1983)		
x_2	0.000292	0.0
x_3	0.9914	1.0
AV	83.54	83.33
V_A	83.43	83.33
MBLOG	0.0244	0.0
MX	3.341	3.333
MXS	3.339	3.333
M	2.506	2.500
MS	2.484	2.500
P,R	4.177	4.167
B	10.0	10.0

6. Optimality Analysis of Model 2.1(B)

In model 2.0(B) (the "B" denotes the second type of objective function), each function v_t is of the form

$$
\begin{aligned}
v_t &= w_t f_t(AV_t, MBLOG_t, M_t, MS_t) \\
&= w_t AV_t \min\{h_2(x_{2t}), h_3(x_{3t})\} ,
\end{aligned}
\tag{46}
$$

where w_t, x_{2t}, x_{3t}, h_2, and h_3 were defined at the beginning of Section 5. Thus model 2.0(B) differs from model 2.0(A) in that the product h_2h_3 is now replaced by $\min\{h_2,h_3\}$.

The rationale for the objective function of model 2.0(B), as compared to that of model 2.0(A), is that the minimum of h_2 and h_3 , as opposed to their product, is the appropriate quantity by which to multiply AV_t in order to obtain the effective asset value.

Now insert v_t from (46) into the average value objective function (18) to obtain

$$
V(T) = \frac{1}{T} \sum_{t=1}^{T} w_t AV_t \min\{h_2(x_{2t}), h_3(x_{3t})\} . \tag{47}
$$

As in Section 5, we suppose that the positive weights w_t satisfy $\sum_{t=1}^{T} w_t = T$ and $\lim_{t\to\infty} w_t = 1$, and we assume that $AV_t \to AV$, $x_{2t} \to x_2$, and $x_{3t} \to x_3$ as $t \to \infty$. Then, as $T \to \infty$, $V(T)$ approaches the limit

$$
V_B = V_B(x_2,x_3) = AV \min\{h_2(x_2),h_3(x_3)\} . \tag{48}
$$

Hence another appropriate objective function for steady state model 2.1 is $V_B(x_2,x_3)$, as given by (48), and we henceforth refer to the following optimization problem as model 2.1(B):

$$\text{Maximize} \quad V_B(x_2,x_3) = AV(x_2,x_3)\ \min\{h_2(x_2),h_3(x_3)\}$$

$$\text{subject to} \quad 0 \leqslant x_2 \leqslant a_4/a_{27}\ ,$$

$$0 \leqslant x_3 \leqslant 1\ .$$

Model 2.1(B) can be recast as a relatively simple NLP problem in several different ways; we provide just one such approach here. Let $x_4 = \min\{h_2(x_2),h_3(x_3)\}$. Then model 2.1(B) is equivalent to

$$\text{Maximize} \quad x_4 AV(x_2,x_3)$$

$$\text{subject to} \quad x_4 \leqslant h_2(x_2)\ ,$$

$$x_4 \leqslant h_3(x_3)\ ,$$

$$0 \leqslant x_2 \leqslant a_4/a_{27}\ ,$$

$$0 \leqslant x_3 \leqslant 1\ .$$

We may instead maximize the logarithm of the objective function $x_4AV(x_2,x_3)$, and, letting $x_5 = \log x_4$, we have the equivalent problem

$$\text{Maximize} \quad x_5 + \log AV(x_2,x_3)$$

$$\text{subject to} \quad x_5 \leqslant \log h_2(x_2)\ ,$$

$$x_5 \leqslant \log h_3(x_3)\ ,$$

$$0 \leqslant x_2 \leqslant a_4/a_{27}\ ,$$

$$0 \leqslant x_3 \leqslant 1\ .$$

Even without solving model 2.1(B) we can partly characterize an optimal solution, regardless of the specific forms of h_2 and h_3 . We phrase the result as a lemma.

Lemma:

Let (x_2^*, x_3^*) be an optimal solution to model 2.1(B). Then either (a) $h_2(x_2^*) = h_3(x_3^*)$, or (b) $x_2^* = a_4/a_{27}$ and $h_3(x_3^*) < h_2(x_2^*)$.

Proof: First note that, by (38), AV is a continuous function of (x_2, x_3) , and it is strictly increasing in x_2 and strictly decreasing in x_3 . Now suppose $h_2(x_2^*) < h_3(x_3^*)$. Then $h_3(x_3^*) > 0$, and so $x_3^* > 0$. Hence x_3 may be decreased to $x_3' = x_3^* - \varepsilon > 0$, where $\varepsilon > 0$, so that $AV(x_2^*, x_3') > AV(x_2^*, x_3^*)$ and $h_2(x_2^*) < h_3(x_3')$. Thus (x_2^*, x_3') is a better solution than (x_2^*, x_3^*) , which is a contradiction.

Now suppose $h_3(x_3^*) < h_2(x_2^*)$. If $x_2^* < a_4/a_{27}$, then, by an argument similar to the one above, x_2 may be increased to $x_2' = x_2^* + \varepsilon$ to yield a better solution. The only remaining possibility is that $h_3(x_3^*) < h_2(x_2^*)$ and $x_2^* = a_4/a_{27}$ (and this does not occur with the currently estimated values of the data).

Model 2.1(B) is equivalent to a small and relatively simple NLP problem, and there are definite advantages, to be discussed in Section 7, in solving it by an appropriate NLP code. For the purposes of this paper, however, we solved it by making use of the lemma above. Since $h_2(x_2^*) = h_3(x_3^*)$ it suffices to maximize $AV[x_2, g_3(x_2)]h_2(x_2)$ subject to $0 \leqslant x_2 \leqslant a_4/a_{27}$, where $g_3(x_2)$ expresses x_3 as a function of x_2 from the condition $h_2(x_2) = h_3(x_3) = h_3[g_3(x_2)]$. We solved this optimization problem involving x_2 only using a one-dimensional search and calculating $x_3 = g_3(x_2)$ for each value of x_2 tested.

TABLE 2

SOLUTIONS TO MODEL 2.1(B)

Variable	Optimal Value	Near-optimal Value
(A) Data from Falk and McCormick (1983)		
x_2	0.00111	0.0
x_3	0.9882	1.0
AV	102.10	101.69
V_B	101.89	101.69
MBLOG	0.1133	0.0
MX	4.082	4.068
MXS	4.074	4.068
M	2.552	2.542
MS	2.522	2.542
P,R	3.403	3.390
B	10.0	10.0
(B) Data from Clark (1983)		
x_2	0.00113	0.0
x_3	0.9880	1.0
AV	83.68	83.33
V_B	83.50	83.33
MBLOG	0.0946	0.0
MX	3.345	3.333
MXS	3.336	3.333
M	2.510	2.500
MS	2.480	2.500
P,R	4.184	4.167
B	10.0	10.0

Table 2 presents the optimal solutions to model 2.1(B) for both the data sets of Clark (1983) and Falk and McCormick (1983). As in the solutions to model 2.1(A), the optimal values of x_2 and x_3 are extremely close to 0 and 1, respectively. Also shown in Table 2 are the values of $V_B(x_2 = 0, x_3 = 1)$; these are only about 0.2% less than the optimal values. For practical purposes, therefore, with presently available data the solution $x_2 = 0$ and $x_3 = 1$ is an optimal one for model 2.1(B), as it was found to be for model 2.1(A).

7. Conclusions and Suggestions for Further Study

Comparison of Tables 1 and 2 shows that, with presently available data, models 2.1(A) and 2.1(B) yield essentially the same solution. For practical purposes this solution is $x_2 = 0$ and $x_3 = 1$. This situation will change, however, if certain of the input data change appropriately. For example, we may expect the optimal solutions to depend quite heavily on the functions h_2 and h_3 . As an illustration, we solved models 2.1(A) and 2.1(B) with the data of Falk and McCormick (1983) and p_2 of $h_2(x_2)$ set at 5 instead of 58. This has the effect of increasing considerably the value of $h_2(x_2)$ for $x_2 > 0$ and makes h_2 a mirror image of h_3 . The results are shown in Table 3; the solutions of models 2.1(A) and 2.1(B) are no longer the same, and they yield values of x_2 and x_3 further from 0 and 1, respectively, than in the case $p_2 = 58$.

Falk and McCormick (1983) solved model 2.0(B) [i.e., model 2.0 with an objective function of the form (47)] by using piecewise-linear approximations to h_2 and h_3 . The dynamic model they solved had five time periods and used the initial values $B_0 = 10$, $AV_0 = 100$,

TABLE 3

SOLUTIONS WITH $p_2 = 5$ AND DATA FROM FALK AND McCORMICK (1983)

Model 2.1(A)		Model 2.1(B)	
Variable	Optimal Value	Variable	Optimal Value
x_2	0.0338	x_2	0.0465
x_3	0.9909	x_3	0.9535
AV	105.03	AV	107.30
V_A	103.26	V_B	104.23

$MBLOG_0 = 1$. It is interesting, and encouraging, to note that the optimal solution they obtained agrees quite closely with our solution to steady-state model 2.1(B). We may therefore hope that in many cases the solution to a steady-state model, such as model 2.1(A) or model 2.1(B), will yield both insight into the nature of the solution of a corresponding dynamic model and a good start toward the exact solution of that model.

Both models 2.1(A) and 2.1(B) can be solved as relatively simple NLP problems, and a number of different algorithms and computer codes could be used to solve them this way [see, e.g., McCormick (1983)]. There are two key advantages to such an approach. First, the use of a dependable computer code is likely to guarantee a correct solution and avoid both theoretical and numerical errors that might arise in developing specialized algorithms. This should be important as we proceed to more complex steady-state models, to be discussed below, and it should save time in finding solutions to them. A computer code that works well on a steady-

state problem might then be applied to a dynamic version of that problem in order to find a solution fairly efficiently. Second, sensitivity analysis of NLP problems is now available with certain computer codes, and this would provide an excellent way to determine the sensitivity of the problem optimal solution to the values of the various parameters. For a thorough discussion of this subject see Fiacco (1983).

Clark (1983) also formulates dynamic model 3.0, which has three sets of control variables. One of our next tasks will be an attempt to derive and analyze a steady-state version of this model. We shall also examine asymptotic versions of models 2.0 and 3.0 in which the limiting growth is assumed to be either linear or exponential. We hope to derive limiting models that are analogous to steady-state model 2.1 and then analyze them in conjunction with appropriate objective functions.

Another task that should be undertaken is the rigorous verification of the limiting behavior we have assumed in this paper.

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